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Wave fronts with one principal curvature a constant in the hyperbolic three-space

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Abstract

In this note, we prove that weakly complete wave fronts with one principal curvature a constant c in the hyperbolic 3-space is either a totally umbilical sphere or umbilic free, if $|c|>1$. Moreover, we derive their orientability.

1 Introduction

By the Hartman-Nirenberg theorem, complete flat surfaces in the Euclidean 3-space $\boldsymbol{R}^3$ are cylinders over a complete planar regular curve (cf. [2]). This fact implies that such surfaces are trivial. On the other hand, if we admit some singularities, there exist many nontrivial examples of flat surfaces. Murata-Umehara investigated global properties of flat surfaces with admissible singularities called *flat fronts* and then proved the following (for precise definitions, see Section 2).

Fact 1.1 ([5]). *A complete flat front in the Euclidean 3-space whose singular point set is non-empty has no umbilics, is orientable and co-orientable. Moreover, if its ends are embedded, there exist at least four singular points other than cuspidal edges.*

This estimate is sharp (see FIGURE 1).

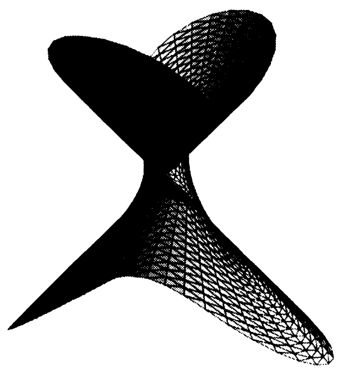

Figure 1: A complete flat front in $\boldsymbol{R}^3$ which has four singular points other than cuspidal edges.

We here remark that a flat surface is considered to be a surface such that one of the principal curvatures is identically zero. In the case of nonzero constant, Shiohama and Takagi [6] showed that *a complete surface one of whose principal* *curvatures is a nonzero constant is either totally umbilical or umbilic-free.* The latter case, such a surface is a tube of a complete regular curve in $\boldsymbol{R}^3$ (i.e., a *channel surface*). In [4], the author investigated wave fronts such that one of the principal curvatures is a nonzero constant (cf. Definition 3.1) and proved the following.

Fact 1.2 ([4]). *A weakly complete wave front in the Euclidean 3-space such that one of the principal curvatures is a nonzero constant has no umbilics and is orientable.*

Although wave fronts with one principal curvature a nonzero constant are co-orientable by definition (cf. Remark 3.2), *there exists co-orientable and non-orientable ones* (see FIGURE 2).

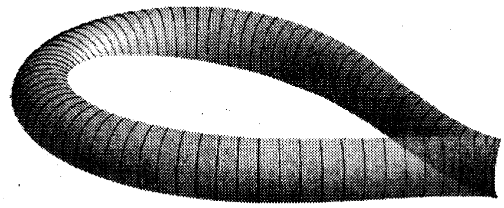

Figure 2: A non-orientable wave front with one principal curvature a nonzero constant in $\boldsymbol{R}^3$.

In the case of non-flat space forms, Aledo-Gálvez [1] investigated (immersed) surfaces with one principal curvature a constant c in the hyperbolic 3-space H^3. In particular, they proved that *a complete surface one of whose principal curvatures is a constant c is either totally umbilical or umbilic-free, if* $|c| > 1$ [1, Theorem 1.1]. Moreover, they showed that, if $|c| \leq 1$, such a result does not hold. That is, if $|c| \leq 1$, they exhibited examples of non-totally-umbilical complete surfaces one of whose principal curvatures is a constant c whose umbilic point set is not empty [1, Example 2.1, Example 2.2]. While their examples are given by the first and second fundamental forms, Izumiya-Saji-Takahashi gave an explicit description of such examples in the case of $|c| = 1$ [3, Example 5.7].

In this paper, we give a generalization of Aledo-Gálvez's Theorem [1, Theorem 1.1] as follows (cf. Theorem 3.7 and Theorem 3.8).

Theorem 1.3. *A weakly complete wave front in the hyperbolic 3-space such that one of the principal curvatures is a constant c satisfying $|c| > 1$ has no umbilics and is orientable.*

This theorem is a direct conclusion of Theorem 3.7 and Theorem 3.8. In the case of $|c| \leq 1$, such a result does not hold (see [1, Example 2.1, Example 2.2]).

This paper is organized as follows. In Section 2, we review fundamental properties of wave fronts in H^3. Then, in Section 3, we define wave fronts one of whose principal curvatures is a constant and give a proof of Theorem 1.3.

2 Preliminaries : wave fronts in H^3

In this section, we review fundamental properties of wave fronts in the hyperbolic 3-space H^3. Here, we regard H^3 as

$$H^3 = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \boldsymbol{R}_1^4\,;\ \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1,\ x_0 > 0\},$$

where $\boldsymbol{R}_1^4$ is the Lorentz-Minkowski 4-space with the inner product

$$\langle \boldsymbol{x}, \boldsymbol{x}\rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2, \qquad \boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \boldsymbol{R}_1^4.$$

If we denote by S_1^3 the de Sitter 3-space $S_1^3 = \{\boldsymbol{x} \in \boldsymbol{R}_1^4\,;\ \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$, the unit tangent bundle T_1H^3 of H^3 is given by

$$T_1H^3 = \{(p, v) \in H^3 \times S_1^3\,;\ \langle p, v\rangle = 0\}.$$

Let M^2 be a smooth 2-manifold and $f : M^2 \to H^3$ be a smooth map. We call f a *frontal*, if for any point $p \in M^2$, there exists a neighborhood U of p and a smooth map $\nu : U \to S_1^3$ such that

$$\langle df_p(\boldsymbol{v}), \nu(p)\rangle = 0$$

holds for all $\boldsymbol{v} \in T_pM^2$. Then, ν is said to be the unit normal vector field of the frontal f. If ν is well-defined on M^2, f is called *co-orientable.* Moreover, f is *orientable* if M^2 is orientable. A point $p \in M^2$ is said to be a *singular* (resp. *regular*) point if $\operatorname{rank}(df)_p < 2$ (resp. $\operatorname{rank}(df)_p = 2$). As in the introduction, we call the frontal f *wave front*, if the map

$$L := (f, \nu) : U \longrightarrow T_1H^3$$

is an immersion. The map L is called the *Legendrian lift* of f.

Lemma 2.1 ([5, Lemma 1.1]). *Let M^2 be a smooth 2-manifold and $f : M^2 \to H^3$ be a co-orientable wave front. If $p \in M^2$ is a singular point of f, then there exist a real number $\delta > 0$ such that p is a regular point of the parallel front $f_\delta := (\cosh\delta)f + (\sinh\delta)\nu$.*

For a co-orientable wave front $f : M^2 \to H^3$, take $p \in M^2$ arbitrary. By Lemma 2.1, there exist a neighborhood U and a real number δ such that f_δ is immersion on U. Then, a point $p \in M^2$ is called *umbilic* of f if p is umbilic point of f_δ. By definition, umbilic points are common in its parallel family.

Lemma 2.2. *Let M^2 be a smooth 2-manifold, $f : M^2 \to H^3$ be a co-orientable wave front and $p \in M^2$ be a singular point of f. Then, p is umbilic if and only if $\operatorname{rank}(df)_p = 0$ holds.*

Lemma 2.2 is an analogue of [4, Lemma 2.2].

Lemma 2.3 ([5, Lemma 1.3]). *Let M^2 be a smooth 2-manifold, $f : M^2 \to H^3$ be a co-orientable wave front and ν be a unit normal vector field of f. For a non-umbilic point $p \in M^2$, there exist a local coordinate system $(U; u, v)$ centered at p such that*

f_u and ν_u (resp. f_v and ν_v) are linearly independent on U. In particular, the pair $\{f_u, \nu_u\}$ (resp. $\{f_v, \nu_v\}$) does not vanishes at the same time and

$$\langle f_u, f_v \rangle = \langle f_u, \nu_v \rangle = \langle f_v, \nu_u \rangle = 0$$

holds.

Such a coordinate system is called *principal curvature line.*

Definition 2.4 (cf. [5, Definition 1.5]). Let M^2 be a smooth 2-manifold and $f : M^2 \to H^3$ be a co-orientable wave front. A direction $\boldsymbol{v} \in T_pM^2$ is called a *principal direction* of f if $df(\boldsymbol{v})$ and $d\nu(\boldsymbol{v})$ are linearly dependent. Moreover, for an open interval $I \subsetneqq \boldsymbol{R}$, a curve $\sigma(t) : I \to M^2$ is called a *principal curvature line* if $\sigma'(t)$ gives a principal direction for all $t \in I$.

On a principal curvature line coordinate neighborhood, every coordinate curve gives a principal curvature line.

For $j = 1, 2$, let $\Lambda_j : M^2 \to P^1(\boldsymbol{R})$ be the *principal curvature map* of a wave front f (for a precise definition, see [5, Section 1]). In particular, if $(U; u, v)$ is a principal curvature line coordinate system, $\Lambda_j|_U : U \to P^1(\boldsymbol{R})\,(j = 1, 2)$ coincide with the smooth maps

$$\Lambda_1 = [-\nu_u : f_u], \qquad \Lambda_2 = [-\nu_v : f_v],$$

respectively. Here, where $[-\nu_u : f_u]$ and $[-\nu_v : f_v]$ mean the proportional ratio of $\{-\nu_u, f_u\}$ and $\{-\nu_v, f_v\}$ respectively as elements of the real projective line $P^1(\boldsymbol{R})$.

Proposition 2.5 ([5, Lemma 1.7]). *Let $f : M^2 \to H^3$ be a co-orientable wave front and Λ_1, Λ_2 be the principal curvature maps of f. Then, a point $p \in M^2$ is umbilic if and only if $\Lambda_1(p) = \Lambda_2(p)$ holds. On the other hand, $p \in M^2$ is a singular point if and only if either $\Lambda_1(p) = [1 : 0]$ or $\Lambda_2(p) = [1 : 0]$ holds.*

At the end of this section, we recall the weakly completeness of wave fronts as follows. Let $f : M^2 \to H^3$ be a wave front and ν be a (locally defined) unit normal vector field of f. Then the symmetric covariant 2-tensor

$$ds^2_{\#} := \langle df, df \rangle + \langle d\nu, d\nu \rangle$$

gives a Riemannian metric on M^2 which is called a *lift metric* of f. The lift metric is a pull-back metric of the Sasakian metric of the unit tangent bundle T_1H^3 of H^3 through the Legendrian lift $L = (f, \nu)$ of f. The lift metric $ds^2_{\#}$ is independent of a choice of ν.

Definition 2.6. A wave front is called *weakly complete* if its lift metric gives a complete Riemannian metric.

3 Wave fronts one of whose principal curvatures is a nonzero constant

In this section, we give a definition of wave fronts one of whose principal curvatures is a nonzero constant. Then, we give a proof of Theorem 1.3 by showing Theorem 3.7 and Theorem 3.8.

3.1 Definitions

Let M^2 be a smooth 2-manifold. Consider a co-orientable front $f : M^2 \to H^3$ such that for some real numbers a, $b \in \boldsymbol{R}$ $(a^2 + b^2 \neq 0)$, f satisfies

$$\text{rank}(a\,(d\nu)_p + b\,(df)_p) < 2 \tag{3.1}$$

for any $p \in M^2$, where $\nu : M^2 \to S^3_1$ is the unit normal vector field of f. If $a \neq 0$, $b = 0$, then f is called an extrinsically flat front, and if $a = 0$, $b \neq 0$, then all the points of M^2 are singular.

From now on, we consider the case $a \neq 0$, $b \neq 0$. Setting $c = b/a$, (3.1) turns out to be

$$\text{rank}((d\nu)_p + c\,(df)_p) < 2 \tag{3.2}$$

for any $p \in M^2$.

Definition 3.1. Let c be a real number, $f : M^2 \to H^3$ be a co-orientable front and $\nu : M^2 \to S^3_1$ be the unit normal vector field of f. Then, f is called *one of whose principal curvatures is a constant* c if f satisfies (3.2).

Remark 3.2 (Non-co-orientable case). Consider a non-co-orientable wave front satisfying (3.2). Changing ν to $-\nu$, we have that such a wave front satisfis both of

$$\text{rank}((d\nu)_p + c\,(df)_p) < 2 \qquad \text{and} \qquad \text{rank}((d\nu)_p - c\,(df)_p) < 2, \tag{3.3}$$

for any $p \in M^2$. (3.3) implies that such a wave front must be isoparametric (i.e., both of the principal curvatures are constant), and hence has no singular points. Since isoparametric surfaces must be orientable, a wave front satisfying (3.3) must be co-orientable. This is a contradiction. Therefore, we have that *wave fronts satisfying* (3.2) *must be co-orientable.*

3.2 Proof of Theorem 1.3

From now on, we denote by $\mathcal{U}_f$ the umbilic point set of a wave front $f : M^2 \to H^3$. Lemma 3.3 and Lemma 3.4 can be proved in the similar way as [4, Lemma 3.5] and [4, Lemma 3.6], respectively.

Lemma 3.3. *Let $f : M^2 \to H^3$ be a wave front one of whose principal curvatures is a constant c. If $p \in M^2$ is a umbilic point of f, the f is regular at p.*

Lemma 3.4. *Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant c and $q \in M^2 \setminus \mathcal{U}_f$ be a non-umbilic point of f. Then there exists a curvatureline coordinate system $(U; u, v)$ around q such that*

- *u-curves are curvature line of Λ_1, v-curves are curvature line of $\Lambda_2 \equiv [c : 1]$,*
- $|f_v| \equiv 1$.
- $\nu_u + cf_u \neq \mathbf{0}, \qquad \nu_v + cf_v = \mathbf{0}, \qquad f_{vv} = f + c\nu$

hold on U, where $\mathbf{0} = (0, 0, 0, 0)$.

A regular curve in H^3 is called a *planar circle*, if its curvature function is a constant greater than 1 and its torsion function is identically zero. For a planar circle $\hat{\sigma} = \hat{\sigma}(t)$, there exist a point $p \in H^3$ such that $\mathrm{dist}_{H^3}(p, \hat{\sigma}(t))$ is a constant for all t, where $\mathrm{dist}_{H^3}(\cdot,\cdot)$ is the distance function of H^3. We call p the center of $\hat{\sigma}$. Lemma 3.5 and Lemma 3.6 can be proved in the similar way as [4, Lemma 3.7] and [4, Lemma 3.8], respectively.

Lemma 3.5. *Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant c and $\sigma(t) : \boldsymbol{R} \supseteqq I \to M^2$ be a principal curvatureline of $\Lambda_2 \equiv c$ parametrized by arc-length passing through a non-umbilic point $q \in M^2 \setminus \mathcal{U}_f$. If $|c| > 1$, $\hat{\sigma}(t) := f \circ \sigma(t)$ is a planar circle in H^3 whose curvature is c and there exist real constants $a, b \in \boldsymbol{R}$ such that Λ_1 is given by*

$$(3.4)\quad \Lambda_1(\sigma(t)) = \Big[1 + c(c^2-1)\left(a\cos\left(\sqrt{c^2-1}t\right) + b\sin\left(\sqrt{c^2-1}t\right)\right) : \\ c + (c^2-1)\left(a\cos\left(\sqrt{c^2-1}t\right) + b\sin\left(\sqrt{c^2-1}t\right)\right)\Big]$$

on $\sigma(t)$. Furthermore, $\sigma(I)$ and $\mathcal{U}_f$ has no intersection.

Lemma 3.6. *Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant c with $|c| > 1$ and $(U; u, v)$ be a curvatureline coordinate system as in Lemma 3.4 around a non-umbilic point $q \in M^2 \setminus \mathcal{U}_f$. Then, the map $C : U \to H^3$ defined by*

$$C(u,v) = \frac{1}{\sqrt{c^2-1}}\left(c\,f(u,v) + \nu(u,v)\right)$$

is independent of v and is a regular curve $C = \gamma(u)$ in H^3. Moreover, if we set $\sigma_{u_0,v_0}(t) : \boldsymbol{R} \supseteqq J \to M^2$ as the curvatureline of Λ_2 such that $\sigma_{u_0,v_0}(0) = (u_0, v_0) \in U$, the center of the planar circle $\hat{\sigma}_{u_0,v_0} := f \circ \sigma_{u_0,v_0}$ is $\gamma(u_0)$ and the image of $\hat{\sigma}_{u_0,v_0}$ is included in the normal plane $\gamma'(u_0)^{\perp}$.

Theorem 3.7. *Let c be a constant satisfying $|c| > 1$ and $f : M^2 \to H^3$ be a wave front one of whose principal curvature is c. If f is weakly complete, f is totally umbilic or umbilic-free. In the latter case, f is described as*

(3.5)
$$f(u,v) = \frac{1}{\sqrt{c^2-1}}\left(-c\gamma(u) + \cos\left(\sqrt{c^2-1}t\right)\boldsymbol{e}_1(u) + \sin\left(\sqrt{c^2-1}t\right)\boldsymbol{e}_2(u)\right),$$

where $(u,v) \in \boldsymbol{R} \times S^1$, $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$, $\gamma(u)$ is a complete regular curve in H^3 and $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a orthonormal frame of the normal bundle of γ.

Proof. Assume that f is not totally umbilic. First of all, we shall prove that the curvatureline of $\Lambda_2 \equiv c$ passing through the non-umbilic point $p \in M^2 \setminus \mathcal{U}_f$ is defined on S^1. Let $(U; u, v)$ be a curvatureline coordinate system around p as in Lemma 3.4. Then each curvatureline of Λ_2 is given by the v-curves on U. The lift metric $ds^2_{\#}$ of f is given by

$$ds^2_{\#} = \langle df, df\rangle + \langle d\nu, d\nu\rangle = (\langle f_u, f_u\rangle + \langle \nu_u, \nu_u\rangle)du^2 + (1+c^2)dv^2$$

on U. In particular, each v-curve is a geodesic of $ds^2_{\#}$, and hence it is defined on $\boldsymbol{R}$ since f is weakly complete. Since the image of each curvatureline of Λ_2 is a planar circle, the domain of each curvatureline is S^1.

Suppose that the umbilic point set $\mathcal{U}_f$ of f is not empty. Take an umbilic point $q \in \partial\mathcal{U}_f$. Then there exists a sequence $\{p_n\} \subsetneqq M^2 \setminus \mathcal{U}_f$ such that $\lim_{n\to\infty} p_n = q$. For each p_n, let σ_n be the curvatureline of Λ_2 passing through p_n. By Lemma 3.5, $\hat{\sigma}_n := f \circ \sigma_n$ is a planar circle of a constant curvature c. Therefore, there exists a subsequence $\{n_k\}$ such that $\hat{\sigma}_q = \lim_{k\to\infty} \hat{\sigma}_{n_k}$ is also a planar circle of a constant curvature c. Every point on the inverse image σ_q of $\hat{\sigma}_q$ through f is umbilic by Lemma 3.5.

On the other hand, by Lemma 3.5, For each $\sigma_{n_k} = \sigma_{n_k}(v)$, there exist v_k such that $\Lambda_1(\sigma_{n_k}(v_k)) = [1 : c]$. If we take the limit as $k \to \infty$, we have $\sigma_q = \lim_{k\to\infty} \sigma_{n_k}$. Therefore, by the continuity of the principal curvature map Λ_1, there exists a point on σ_q such that $\Lambda_1 = [1 : c] \neq [c : 1] = \Lambda_2$, which is a contradiction. Thus we have $\mathcal{U}_f = \emptyset$. □

Theorem 3.8. *Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant c with $|c| > 1$. If f is weakly complete, f is orientable.*

Proof. If f is totally umbilic, f is orientable. Thus we assume that f is not totally umbilic. Then, by Theorem 3.7, f is represented as in (3.5). Take an orthonormal frame $\boldsymbol{e}_1, \boldsymbol{e}_2$ of γ such that $\{\gamma'(u), \boldsymbol{e}_1(u), \boldsymbol{e}_2(u)\}$ is a positively oriented orthogonal frame. Setting $\boldsymbol{e}_0 := \boldsymbol{e}_1 \times \boldsymbol{e}_2$, we have

$$\gamma'(u) = \varphi(u)\,\boldsymbol{e}_0(u), \qquad \left(\varphi(u) = \sqrt{\langle \gamma'(u), \gamma'(u)\rangle}\right).$$

If f is not orientable, there exist real numbers u_0, L such that $\gamma(u+L) = \gamma(u)$ holds for each $u \in \boldsymbol{R}$ and

$$\boldsymbol{e}_1(u_0 + L) \times \boldsymbol{e}_2(u_0 + L) = -\boldsymbol{e}_1(u_0) \times \boldsymbol{e}_2(u_0)$$

holds. Since $\boldsymbol{e}_0(u_0 + L) = -\boldsymbol{e}_0(u_0)$, we have

$$\gamma'(u_0 + L) = \varphi(u_0 + L)\,\boldsymbol{e}_0(u_0 + L) = -\varphi(u_0)\,\boldsymbol{e}_0(u_0) = -\gamma'(u_0),$$

which contradicts to $\gamma'(u_0 + L) = \gamma'(u_0)$. □

Theorem 3.7 and Theorem 3.8 imply Theorem 1.3.

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